EVERYWHERE CONTINUOUS NOWHERE DIFFERENTIABLE FUNCTIONS

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Abstract. Here I discuss the use of everywhere continuous nowhere di erentiable functions, as well as the proof of an example of such a function. First, I will

the left and the slope as conceived from the right. The logical choice is to not choose between them and to declare the derivative non-existent, as is done formally.

The deception occurs in that for continuous functions failure of di erentiation dense in R [1]. Without exposure to such functions, continuity can become conflated with \easy to draw", and even the example above is not continuous at countably many points.

The derivative has even deeper connotations of smoothness. In order for a function to have a derivative at a point c, not only must the function be continuous at c and hence providing all of the connotations related to that, but one must be able to approximate the function linearly at that point. Because linear functions are particularly smooth, the existence of such an approximation hints that the original function must be smooth as well. In fact, the formula for the derivative provides a formal method for finding such an approximation by taking the limit of the slope between c and some other point as the point approaches c. When drawing

Moreover, we have $h(x + 2n) = h(x)$ for all n 2 N. For $n = 1$ this is just h(x + 2) just

If $f_i(x) = 0$ for some i 2 N, then $2^i x$ 2 2Z. $f_{i+1}(x) = \frac{f_i(2x)}{2}$, and 2 $2^i x$ will also be in 2Z, so $f i + 1(x) = 0$. Therefore, given an n such that $f n + 1(x) = 0$, we have that $g(x) = g_n(x)$. Using this we can say that because $f_1(0) = h(2(0)) = h(2) = 0$, $g(0) = g_0(0) = h(0) = 0$. Moreover, for x 2 $[-\frac{1}{2^n}; \frac{1}{2^n}]$, $f_n(x) = jxj$. Also, g will have a period of 2 just as h did, because for all i 2 N, $f_i(x + 2) = f_i(x)$. f_0 is just h for which this is assumed. Assuming inductively that $f_i(x + 2) = f(x)$ for some i 2 N, we know that $f_{i+1}(x) = \frac{f_i(2x)}{2}$. Therefore,

$$\begin{aligned} f_{i+1}(x+2) &= \frac{f_i(2x+4)}{2} \\ &= \frac{f_i(2x+2)}{2} \\ &= \frac{f i(2x)}{2} \\ &= f_{i+1}(x) \end{aligned}$$

Hence, $f_i(x + 2) = f(x)$ holds for all i 2 N by induction1.3826 Td[(f)-108 (i)]TJ/Tapply Tf0.942 0 Tdf6.(x1 Tf( )Tj/f(StyleSp
hold0/755 1 Tfi( )Tj/T1_4 1 Tf2.

for some $L \in \mathbb{R}$. However, $2^{m+1} \cdot \frac{1}{2^m} \in 2\mathbb{Z}$, so $f_{m+1}(x_m) = 0$ and thus $g(x_m) = g_m(x_m)$. Therefore,

$$
\begin{aligned}
2^m g(2^{-m}) &= 2^m g_m(2^{-m}) \\
&= 2^m \sum_{i=0}^{m} f_i(2^{-m}) \\
&= 2^m \sum_{i=0}^{m} 2^{-m} \\
&= \sum^{m} 1
\end{aligned}
$$

Now consider the sequence $x_m = 1 - \frac{1}{2^{m+1}}$. First, $2^{m+2}x_m = 2^{m+2} - 2 \; 2 \; 2Z$, so $f_{m+2}(x_m) = 0$, and thus $g(x_m) = g_{m+1}(x_m)$. Now, $\lim x_m = 1$, so if the derivative of g exists at 1 it will equal,

$$\lim \frac{g(x_m) - g(1)}{x_m - 1} = \lim \frac{g(1 - \frac{1}{2^{m+1}}) - 1}{-2^{-m-1}}$$

$$= \lim -2^{m+1} \; {}^{m+1}X$$

Finally, we we look at points x 2 R that do not have this form. For each N 2 N, there will exist a z_N 2 Z such that $\frac{z_N}{2^N} < x < \frac{z_N+1}{2^N}$. As such we can create two sequences (x_m) and (y_m) out of these where $xm = \frac{z_m}{2^m}$ and $ym = \frac{z_m+1}{2^m}$, where $\lim x_m = x = \lim y_m$. In fact, these describe the end points of the line on which x sits in f_m, and that line will be contained in a line on f_n for all $n \leq m$. Therefore, the portion between x_m and y_m

which

0

function k, such as $\sin(3x)$, is barely distinguishable from k, while adding f_3 produces clear changes. Hence, while one typically thinks that a functions derivative is clearly understandable from picture and that having an smooth graph makes a function likely to be di erentiable, this is not necessarily the case.

This proof also demonstrates a way that a function can fail to have a derivative at a point without showing a sharp turn. For example, the slopes of the secant lines could tend towards infinity as the thenot a t

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References

[1] S. Abbot. *Understanding Analysis*. Springer Science + Business Media, LLC, New York (2000).

[2] J. Thim. *Continuous Nowehre Di erentiable Equations*. Master's Thesis. Lulea University of Technology (2003).